

Conditional computability of real functions with respect to a class of operators

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Abstract

For any class of operators which transform unary total functions in the set of natural numbers into functions of the same kind, we define what it means for a real function to be uniformly computable or conditionally computable with respect to this class. These two computability notions are natural generalizations of certain notions introduced in a previous paper co-authored by Andreas Weiermann and in another previous paper by the same authors, respectively. Under certain weak assumptions about the class in question, we show that conditional computability is preserved by substitution, that all conditionally computable real functions are locally uniformly computable, and that the ones with compact domains are uniformly computable. The introduced notions have some similarity with the uniform computability and its non-uniform extension considered by Katrin Tent and Martin Ziegler, however, there are also essential differences between the conditional computability and the non-uniform computability in question.

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1. Introduction

In the paper [7], a notion of uniform computability of real functions was introduced, namely, when a class \mathcal{F} of total functions in \mathbb{N} is given, some real functions were called uniformly \mathcal{F} -computable. The definition of the notion was in the spirit of the approach to computability of real functions originating from [1, 3] and nowadays indicated by the abbreviation TTE (cf., for instance, the monograph [9]). This approach uses (necessarily infinitistic) naming systems for the real numbers and defines the computability of a real function as the existence of some effective procedure which transforms arbitrary names of the arguments into a name of the corresponding function value. The class of the computable real functions may depend on the choice of the naming system and on the sort of effective procedures which are admitted. However, there

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are some choices that hopefully produce the most general intuitively reasonable notion of computability for real functions (we will use the term *TTE computability* for this notion). Such a choice is, for instance, naming the real numbers by sequences of rational numbers converging to them with a given polynomial or exponential rate and transformation of the names through recursive operators (or, as for instance in [9], through oracle Turing machines). As indicated in [4], the restriction to general recursive operators leads to a narrower notion in the case of real functions which are not everywhere defined. Further reduction of the class of operators for the transformation of the names could additionally reduce the corresponding class of computable real functions, and this could be useful for introducing some subrecursive computability notions for real functions.

In the case of uniform \mathcal{F} -computability, real numbers are named (up to technical details) by sequences of rational numbers converging to them with a linear rate, and the transformation of the names is performed by so-called \mathcal{F} -substitutional operators. Roughly speaking, the values of the image of a tuple of functions under such an operator are computed through evaluation of a term built from a variable, ranging over \mathbb{N} , by means of symbols for the functions in the tuple and for functions from \mathcal{F} . (In general, since \mathcal{F} could contain some non-computable functions, the corresponding procedure of transformation of names may be non-effective, and some uniformly \mathcal{F} -computable real functions may turn out to be not TTE computable. However, this cannot happen if \mathcal{F} consists of recursive functions.)

The main attention in [7] is paid to the case when \mathcal{F} is a rather small subrecursive class, namely the class \mathcal{M}^2 (up to the argumentless constants $0, 1, 2, \dots$, it consists of all functions in \mathbb{N} which can be obtained from the successor function, the function $\lambda xy.x \div y$, the multiplication function and the projection functions by finitely many applications of substitution and bounded least number operation). Somewhat surprisingly, the results from [7] easily imply that all elementary functions of calculus are uniformly \mathcal{M}^2 -computable on the compact subsets of their domains. As to the uniform \mathcal{M}^2 -computability of these functions on their whole domains, however, there is a serious obstacle for many of them, since any uniformly \mathcal{M}^2 -computable real function is bounded by some polynomial.

In the paper [6], we introduced a wider notion of \mathcal{F} -computability called conditional \mathcal{F} -computability. Its definition ensures that all conditionally \mathcal{F} -computable real functions are TTE computable in the case when \mathcal{F} consists of recursive functions. Under some weak assumptions on \mathcal{F} , we proved that conditional \mathcal{F} -computability is preserved by substitution, all conditionally \mathcal{F} -computable real functions are locally uniformly \mathcal{F} -computable and all conditionally \mathcal{F} -computable real functions with compact domains are uniformly \mathcal{F} -computable. Moreover, we prove that all elementary functions of calculus (considered on their whole domains) are conditionally \mathcal{M}^2 -computable.¹ We also show the existence of TTE computable real functions which are not conditionally \mathcal{F} -computable, whatever be the class \mathcal{F} .

The supplementary feature of conditional \mathcal{F} -computability in comparison to the uniform one can be informally described as follows. It is now allowed the transformation of the names of the argument values (which produces a name for the corresponding function value) to depend on

¹The proof in [6] of the last statement needs some refinement concerning the functions $\lambda \xi. \sqrt[n]{\xi}$, $n = 2, 3, \dots$. Although the exponential function and the logarithmic function are proved in that paper to be conditionally \mathcal{M}^2 -computable, the expression for $\sqrt[n]{\xi}$ through them does not prove the conditional \mathcal{M}^2 -computability of the functions $\lambda \xi. \sqrt[n]{\xi}$ on their whole domains (including 0 and also the negative real numbers in the case when n is odd). The functions in question are actually uniformly \mathcal{M}^2 -computable. This is proved for the case $n = 2$ in [7] and the proof can be easily modified to encompass the other values of n .

an additional parameter whose value is some natural number. Some term of the sort mentioned above must exist, such that whenever some names of the argument values are given, this number can be found by means of a search until the term in question vanishes (no restriction is imposed on the means used for organizing the search itself).² In order to take into account the parameter's value, we somewhat enlarged the class of procedures used for the transformation of names, and, roughly speaking, this corresponds to realizing them by using terms built in the above-mentioned way from two variables, ranging over \mathbb{N} , instead of one.

The present paper is devoted to a generalization of a part of the considerations in [6] to a situation when the class of the \mathcal{F} -substitutional operators is replaced with an arbitrary class \mathbf{O} of operators in the set of the total unary functions in \mathbb{N} . The dependence on the value of the additional parameter in the case of conditional computability is now realized by adding the corresponding constant function as an additional argument of the operators.

It seems that the approach described above has to do not only with the computational complexity of a real function, but also with the complexity of its definition. There is a more resource-oriented approach to computability of real functions due to Ker-I Ko in [2], where the complexity of computable real functions is connected with discrete polynomial complexity theory. The links between our approach and Ko's one are yet to be specified. At first glance, there is an essential difference between the two approaches, namely Ko's approximations of the real numbers are defined to be of exponential rate, but our approximations are necessarily of polynomial rate for the case, when we compute by means of functions in \mathbb{N} , which are bounded by polynomial.

2. Appropriate classes of operators and their relation to \mathcal{F} -substitutional mappings

As we did in [6], for any $m \in \mathbb{N}$, we will denote by \mathbb{T}_m the set of all m -argument total functions in \mathbb{N} , and for any subset \mathcal{F} of $\bigcup_{m \in \mathbb{N}} \mathbb{T}_m$ and any $k, m \in \mathbb{N}$, we may consider the notion of \mathcal{F} -substitutional mapping of \mathbb{T}_1^k into \mathbb{T}_m . We will be interested in the last notion mainly for the case $m = 1$, and, in particular, we will indicate a way to reduce the case $m > 1$ to this one.

In the present paper, the term "operator" will be used in the following restricted sense. For any $k \in \mathbb{N}$, the mappings of \mathbb{T}_1^k into \mathbb{T}_1 (that is the k -ary operations in \mathbb{T}_1) will be called k -ary operators. The k -ary operators for all $k \in \mathbb{N}$ will generally be called operators.

For any natural number c , let \check{c} be the constant function from \mathbb{T}_1 with value c . The identity function in \mathbb{N} will be denoted by $\text{id}_{\mathbb{N}}$. We set $\mathbb{T} = \bigcup_{m \in \mathbb{N}} \mathbb{T}_m$. We introduce the notion of appropriate class of operators, which, roughly speaking, captures some necessary substitutional properties, possessed by the class of the \mathcal{F} -substitutional operators (as seen from Lemma 2.3).

²In the paper [8], two notions of computability of real functions with respect to some class \mathcal{F} of total functions in \mathbb{N} are introduced and studied, namely the notion of a real function to be uniformly in \mathcal{F} and the wider notion of a real function to be in \mathcal{F} . The definitions of these notions use rational approximations of real numbers more directly than it is done in the TTE approach, but nevertheless there is a similarity to them of the uniform and the conditional computability, respectively. As seen from [5], the similarity of the uniform computability to the first notion is not a superficial one. However, the resemblance between conditional computability and the second notion is not so deep – although the definition of this notion allows a dependence of the approximation process on an additional parameter, the description of its value uses the distance to the complement of the domain of the function, and the means provided by \mathcal{F} could be not sufficient for checking if a given number is appropriate as a value of this parameter (a confusing feature of this notion is that some real functions with complicated domains may turn out to be in the class of the recursive functions without being TTE computable).

Definition 2.1. Let \mathbf{O} be a class of operators. The class \mathbf{O} will be called appropriate, whenever the following conditions are satisfied:

1. For any $k \in \mathbb{N}$ and any $i \in \{1, \dots, k\}$, the operator F defined by means of the equality $F(f_1, \dots, f_k) = f_i$ belongs to \mathbf{O} .
2. The operator F defined by $F(f_1, f_2)(n) = f_1(f_2(n))$ belongs to \mathbf{O} .
3. For any $k, l \in \mathbb{N}$, if F is a k -ary operator belonging to \mathbf{O} , and G_1, \dots, G_k are l -ary operators belonging to \mathbf{O} , the operator H defined by

$$H(g_1, \dots, g_l) = F(G_1(g_1, \dots, g_l), \dots, G_k(g_1, \dots, g_l))$$

also belongs to \mathbf{O} .

4. For any $k \in \mathbb{N}$ and any $(k + 1)$ -ary operator F belonging to \mathbf{O} , the operator G defined by

$$G(f_1, \dots, f_k)(n) = F(f_1, \dots, f_k, \check{n})(n)$$

also belongs to \mathbf{O} .

For the sake of convenience, we recall the definition from [6] of the notion of \mathcal{F} -substitutional mapping.

Definition 2.2. Let $\mathcal{F} \subseteq \mathbb{T}$. For any $k, m \in \mathbb{N}$, certain mappings of \mathbb{T}_1^k into \mathbb{T}_m will be called \mathcal{F} -substitutional. We proceed by induction:

1. For any m -argument projection function h in \mathbb{N} the mapping F defined by means of the equality $F(f_1, \dots, f_k) = h$ is \mathcal{F} -substitutional.
2. For any $i \in \{1, \dots, k\}$, if F_0 is an \mathcal{F} -substitutional mapping of \mathbb{T}_1^k into \mathbb{T}_m then so is the mapping F defined by means of the equality

$$F(f_1, \dots, f_k)(n_1, \dots, n_m) = f_i(F_0(f_1, \dots, f_k)(n_1, \dots, n_m)).$$

3. For any natural number r and any r -argument function f from \mathcal{F} , if F_1, \dots, F_r are \mathcal{F} -substitutional mappings of \mathbb{T}_1^k into \mathbb{T}_m then so is the mapping F defined by means of the equality

$$F(f_1, \dots, f_k)(n_1, \dots, n_m) = f(F_1(f_1, \dots, f_k)(n_1, \dots, n_m), \dots, F_r(f_1, \dots, f_k)(n_1, \dots, n_m)).$$

According to the terminology adopted in this paper, the \mathcal{F} -substitutional mappings of \mathbb{T}_1^k into \mathbb{T}_1 will be called k -ary \mathcal{F} -substitutional operators.

Lemma 2.3. Let $\mathcal{F} \subseteq \mathbb{T}$ and let \mathbf{O} be the class of all \mathcal{F} -substitutional operators. Then \mathbf{O} is appropriate.

Proof. Condition 1 on \mathbf{O} is satisfied by clauses 1 and 2 of Definition 2.2 (considered for the case $m = 1$, when it coincides with the corresponding Definition 6 from [7]). Condition 2 on \mathbf{O} follows from condition 1 on \mathbf{O} and clause 2 of Definition 2.2. Condition 3 on \mathbf{O} is the case $m = 1$ of Proposition 2 in [6]. To prove condition 4 on \mathbf{O} we use induction on the construction of the operator F .

If F is \mathcal{F} -substitutional by clause 1 of Definition 2.2, then so is G .

Suppose that F has the form from clause 2 of Definition 2.2, that is

$$F(f_1, \dots, f_k, f_{k+1})(n) = f_i(F_0(f_1, \dots, f_k, f_{k+1})(n))$$

for some $i \in \{1, \dots, k+1\}$ and, by the inductive hypothesis, the operator F_0 has the considered property. If $i \leq k$, then

$$G(f_1, \dots, f_k)(n) = F(f_1, \dots, f_k, \check{n})(n) = f_i(F_0(f_1, \dots, f_k, \check{n})(n)),$$

so by clause 2 of Definition 2.2, G is \mathcal{F} -substitutional. If $i = k+1$, then

$$G(f_1, \dots, f_k)(n) = F(f_1, \dots, f_k, \check{n})(n) = \check{n}(F_0(f_1, \dots, f_k, \check{n})(n)) = n,$$

so by clause 1 of Definition 2.2, G is \mathcal{F} -substitutional.

Finally, suppose that F is defined by

$$F(f_1, \dots, f_k, f_{k+1})(n) = f(F_1(f_1, \dots, f_k, f_{k+1})(n), \dots, F_r(f_1, \dots, f_k, f_{k+1})(n)),$$

where $f : \mathbb{N}^r \rightarrow \mathbb{N}$ belongs to \mathcal{F} (clause 3 from Definition 2.2) and by the inductive hypothesis the operators F_1, \dots, F_r have the considered property. Then

$$G(f_1, \dots, f_k)(n) = F(f_1, \dots, f_k, \check{n})(n) = f(F_1(f_1, \dots, f_k, \check{n})(n), \dots, F_r(f_1, \dots, f_k, \check{n})(n)),$$

so G is \mathcal{F} -substitutional by clause 3 of Definition 2.2. \square

Lemma 2.4. *Let $\mathcal{F} \subseteq \mathbb{T}$ and $F : \mathbb{T}_1^k \rightarrow \mathbb{T}_{m+1}$ for some $k, m \in \mathbb{N}$, $m \neq 0$. The mapping F is \mathcal{F} -substitutional iff there exists an \mathcal{F} -substitutional mapping $G : \mathbb{T}_1^{k+1} \rightarrow \mathbb{T}_m$, such that*

$$F(f_1, \dots, f_k)(s, t_1, \dots, t_m) = G(f_1, \dots, f_k, \check{s})(t_1, \dots, t_m) \quad (1)$$

for all $f_1, \dots, f_k \in \mathbb{T}_1$ and all $s, t_1, \dots, t_m \in \mathbb{N}$.

Proof. (\Leftarrow). By induction on the construction of G , we will show that for any \mathcal{F} -substitutional mapping $G : \mathbb{T}_1^{k+1} \rightarrow \mathbb{T}_m$, the mapping $F : \mathbb{T}_1^k \rightarrow \mathbb{T}_{m+1}$, defined by the equality (1) is \mathcal{F} -substitutional.

If G is \mathcal{F} -substitutional by clause 1 of Definition 2.2, then so is F .

Let G have the form

$$G(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m) = f_i(G_0(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m))$$

for some $i \in \{1, \dots, k+1\}$ and let by the inductive hypothesis G_0 possess the required property. If $i \leq k$, then

$$F(f_1, \dots, f_k)(s, t_1, \dots, t_m) = f_i(G_0(f_1, \dots, f_k, \check{s})(t_1, \dots, t_m)),$$

so F is \mathcal{F} -substitutional by clause 2 of Definition 2.2. If $i = k+1$, then

$$F(f_1, \dots, f_k)(s, t_1, \dots, t_m) = \check{s}(G_0(f_1, \dots, f_k, \check{s})(t_1, \dots, t_m)) = s,$$

so F is \mathcal{F} -substitutional by clause 1 of Definition 2.2.

Finally, suppose that G is defined by

$$\begin{aligned} G(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m) \\ = f(G_1(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m), \dots, G_r(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m)) \end{aligned}$$

for a function $f : \mathbb{N}^r \rightarrow \mathbb{N}$ belonging to \mathcal{F} (clause 3 of Definition 2.2) and mappings G_1, \dots, G_r , which by the inductive hypothesis possess the considered property. Then

$$F(f_1, \dots, f_k)(s, t_1, \dots, t_m) = f(G_1(f_1, \dots, f_k, \mathfrak{s})(t_1, \dots, t_m), \dots, G_r(f_1, \dots, f_k, \mathfrak{s})(t_1, \dots, t_m)),$$

so F is \mathcal{F} -substitutional by clause 3 of Definition 2.2.

(\Rightarrow). By induction on the construction of F , we will show that for any \mathcal{F} -substitutional mapping $F : \mathbb{T}_1^k \rightarrow \mathbb{T}_{m+1}$, there exists an \mathcal{F} -substitutional mapping $G : \mathbb{T}_1^{k+1} \rightarrow \mathbb{T}_m$, such that the equality (1) holds for all $f_1, \dots, f_k \in \mathbb{T}_1$ and all $s, t_1, \dots, t_m \in \mathbb{N}$.

Suppose $F(f_1, \dots, f_k)(s, t_1, \dots, t_m) = t_i$ for some $i \in \{1, \dots, m\}$. Then the mapping G , defined by $G(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m) = t_i$ satisfies the required condition.

If $F(f_1, \dots, f_k)(s, t_1, \dots, t_m) = s$, then the mapping G , defined by

$$G(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m) = f_{k+1}(t_1)$$

is \mathcal{F} -substitutional (by clauses 1 and 2 of Definition 2.2) and

$$G(f_1, \dots, f_k, \mathfrak{s})(t_1, \dots, t_m) = \mathfrak{s}(t_1) = s = F(f_1, \dots, f_k)(s, t_1, \dots, t_m).$$

If F has the form

$$F(f_1, \dots, f_k)(s, t_1, \dots, t_m) = f_i(F_0(f_1, \dots, f_k)(s, t_1, \dots, t_m))$$

for some $i \in \{1, \dots, k\}$ and G_0 is the \mathcal{F} -substitutional mapping, which exists by the inductive hypothesis for the mapping F_0 , then we can define a mapping G by

$$G(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m) = f_i(G_0(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m)),$$

which is \mathcal{F} -substitutional by clause 2 of Definition 2.2. It follows that

$$F(f_1, \dots, f_k)(s, t_1, \dots, t_m) = f_i(G_0(f_1, \dots, f_k, \mathfrak{s})(t_1, \dots, t_m)) = G(f_1, \dots, f_k, \mathfrak{s})(t_1, \dots, t_m).$$

Finally, suppose that F is defined by

$$F(f_1, \dots, f_k)(s, t_1, \dots, t_m) = f(F_1(f_1, \dots, f_k)(s, t_1, \dots, t_m), \dots, F_r(f_1, \dots, f_k)(s, t_1, \dots, t_m)),$$

where $f : \mathbb{N}^r \rightarrow \mathbb{N}$ belongs to \mathcal{F} (clause 3 from Definition 2.2) and by the inductive hypothesis the mappings F_1, \dots, F_r have the considered property, that is there exist \mathcal{F} -substitutional mappings G_1, \dots, G_r , such that

$$F_i(f_1, \dots, f_k)(s, t_1, \dots, t_m) = G_i(f_1, \dots, f_k, \mathfrak{s})(t_1, \dots, t_m)$$

for all $i \in \{1, \dots, r\}$. Let us define

$$\begin{aligned} G(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m) \\ = f(G_1(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m), \dots, G_r(f_1, \dots, f_k, f_{k+1})(t_1, \dots, t_m)). \end{aligned}$$

Then G is \mathcal{F} -substitutional by clause 3 of Definition 2.2 and

$$\begin{aligned} F(f_1, \dots, f_k)(s, t_1, \dots, t_m) &= f(G_1(f_1, \dots, f_k, \check{s})(t_1, \dots, t_m), \dots, G_r(f_1, \dots, f_k, \check{s})(t_1, \dots, t_m)) \\ &= G(f_1, \dots, f_k, \check{s})(t_1, \dots, t_m). \quad \square \end{aligned}$$

Corollary 2.5. *Let $\mathcal{F} \subseteq \mathbb{T}$ and $F : \mathbb{T}_1^k \rightarrow \mathbb{T}_{m+1}$ for some $k, m \in \mathbb{N}$. The mapping F is \mathcal{F} -substitutional iff there exists an \mathcal{F} -substitutional operator $G : \mathbb{T}_1^{k+m} \rightarrow \mathbb{T}_1$, such that*

$$F(f_1, \dots, f_k)(s_1, \dots, s_m, t) = G(f_1, \dots, f_k, \check{s}_1, \dots, \check{s}_m)(t)$$

for all $f_1, \dots, f_k \in \mathbb{T}_1$ and all $s_1, \dots, s_m, t \in \mathbb{N}$.

For any function $f : \mathbb{N}^k \rightarrow \mathbb{N}$ we define the k -ary operator \mathring{f} by the equality

$$\mathring{f}(f_1, \dots, f_k)(n) = f(f_1(n), \dots, f_k(n))$$

for all $n \in \mathbb{N}$.

Definition 2.6. *Let \mathbf{O} be a class of operators. A function $f : \mathbb{N}^k \rightarrow \mathbb{N}$ will be called representable in \mathbf{O} or \mathbf{O} -representable, for short, if the corresponding operator \mathring{f} belongs to \mathbf{O} .*

Lemma 2.7. *Let $\mathcal{F} \subseteq \mathbb{T}$ and \mathbf{O} be the class of all \mathcal{F} -substitutional operators. Then all functions from \mathcal{F} are \mathbf{O} -representable.*

Proof. We use clause 3 in Definition 2.2 and condition 1 in Definition 2.1, making use of the fact that \mathbf{O} is appropriate. \square

Lemma 2.8. *Let \mathbf{O} be an appropriate class of operators. Then all projection functions in \mathbb{N} are \mathbf{O} -representable.*

Proof. Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be a projection function, that is for some $i \in \{1, \dots, k\}$, $f(n_1, \dots, n_k) = n_i$. Then $\mathring{f}(f_1, \dots, f_k)(n) = f(f_1(n), \dots, f_k(n)) = f_i(n)$ for all $n \in \mathbb{N}$, that is $\mathring{f}(f_1, \dots, f_k) = f_i$ and by condition 1 in Definition 2.1, $\mathring{f} \in \mathbf{O}$. It follows that f is \mathbf{O} -representable. \square

Lemma 2.9. *Let \mathbf{O} be an appropriate class of operators. Then the class of all \mathbf{O} -representable functions is closed under substitution.*

Proof. Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ and $g_i : \mathbb{N}^l \rightarrow \mathbb{N}$, $i = 1, \dots, k$ be \mathbf{O} -representable functions and the function $g : \mathbb{N}^l \rightarrow \mathbb{N}$ be defined by

$$g(n_1, \dots, n_l) = f(g_1(n_1, \dots, n_l), \dots, g_k(n_1, \dots, n_l)).$$

By condition 3 in Definition 2.1, it is sufficient to show that

$$\mathring{g}(f_1, \dots, f_l) = \mathring{f}(\mathring{g}_1(f_1, \dots, f_l), \dots, \mathring{g}_k(f_1, \dots, f_l))$$

holds for all $f_1, \dots, f_l \in \mathbb{T}_1$. This can be seen as follows: for all $n \in \mathbb{N}$ and $f_1, \dots, f_l \in \mathbb{T}_1$,

$$\begin{aligned} \mathring{g}(f_1, \dots, f_l)(n) &= g(f_1(n), \dots, f_l(n)) = f(g_1(f_1(n), \dots, f_l(n)), \dots, g_k(f_1(n), \dots, f_l(n))) \\ &= f(\mathring{g}_1(f_1, \dots, f_l)(n), \dots, \mathring{g}_k(f_1, \dots, f_l)(n)) = \mathring{f}(\mathring{g}_1(f_1, \dots, f_l), \dots, \mathring{g}_k(f_1, \dots, f_l))(n). \quad \square \end{aligned}$$

Lemma 2.10. *Let \mathbf{O} be an appropriate class of operators, and k be a natural number. Then:*

1. *The k -ary operator F defined by $F(f_1, \dots, f_k) = \text{id}_{\mathbb{N}}$ belongs to \mathbf{O} .*
2. *For any $i \in \{1, \dots, k\}$, if F_0 is a k -ary operator belonging to \mathbf{O} then so is the operator F defined by means of the equality $F(f_1, \dots, f_k)(n) = f_i(F_0(f_1, \dots, f_k)(n))$.*
3. *For any natural number r and any r -argument \mathbf{O} -representable function f , if F_1, \dots, F_r are k -ary operators belonging to \mathbf{O} then so is the operator F defined by means of the equality $F(f_1, \dots, f_k)(n) = f(F_1(f_1, \dots, f_k)(n), \dots, F_r(f_1, \dots, f_k)(n))$.*

Proof. If F is the operator from statement 1 of the lemma then, for all $f_1, \dots, f_k \in \mathbb{T}_1$ and all $n \in \mathbb{N}$,

$$F(f_1, \dots, f_k)(n) = n = F_0(f_1, \dots, f_k, \check{n})(n),$$

where the $(k+1)$ -ary operator F_0 is defined by $F_0(f_1, \dots, f_k, f_{k+1}) = f_{k+1}$, hence $F \in \mathbf{O}$ by conditions 1 and 4 of Definition 2.1.

If F is defined in the way from statement 2 then, for all $f_1, \dots, f_k \in \mathbb{T}_1$,

$$F(f_1, \dots, f_k) = F_1(f_i, F_0(f_1, \dots, f_k)),$$

where F_1 is the operator from condition 2 of Definition 2.1, hence $F \in \mathbf{O}$ by conditions 1, 2 and 3 of Definition 2.1.

Finally, suppose that F is defined in the way from statement 3. Then, for all $f_1, \dots, f_k \in \mathbb{T}_1$,

$$F(f_1, \dots, f_k) = \hat{f}(F_1(f_1, \dots, f_k), \dots, F_r(f_1, \dots, f_k)),$$

hence $F \in \mathbf{O}$ by condition 3 of Definition 2.1 and the fact that f is \mathbf{O} -representable. \square

Corollary 2.11. *Let \mathbf{O} be an appropriate class of operators, k be a natural number, and f be a unary \mathbf{O} -representable function. Then the operator F defined by $F(f_1, \dots, f_k) = f$ belongs to \mathbf{O} .*

Proof. Let F_1 be the operator defined by $F_1(f_1, \dots, f_k) = \text{id}_{\mathbb{N}}$. Then, for all $f_1, \dots, f_k \in \mathbb{T}_1$ and all $n \in \mathbb{N}$, $F(f_1, \dots, f_k)(n) = f(F_1(f_1, \dots, f_k)(n))$, hence $F \in \mathbf{O}$ by statements 1 and 3 of Lemma 2.10. \square

Corollary 2.12. *Let \mathbf{O} be an appropriate class of operators, and \mathcal{F} be the class of all \mathbf{O} -representable functions. Then all \mathcal{F} -substitutional operators belong to \mathbf{O} .*

Proof. By induction making use of statements 1, 2 and 3 of Lemma 2.10. \square

The continuity notion for operators is defined in the usual way.

Definition 2.13. *Let $F : \mathbb{T}_1^k \rightarrow \mathbb{T}_1$. The operator F will be called continuous, if the following condition is satisfied: for all $f_1, \dots, f_k \in \mathbb{T}_1$ and $n \in \mathbb{N}$, there exists $u \in \mathbb{N}$, such that*

$$F(g_1, \dots, g_k)(n) = F(f_1, \dots, f_k)(n), \tag{2}$$

whenever $g_1, \dots, g_k \in \mathbb{T}_1$ and $g_1(t) = f_1(t), \dots, g_k(t) = f_k(t)$ for all $t \leq u$.

Lemma 2.14. *For any $\mathcal{F} \subseteq \mathbb{T}$, all \mathcal{F} -substitutional operators are continuous.*

Proof. One shows by induction on the construction of F that every \mathcal{F} -substitutional operator F is continuous. \square

Clearly, the class of all operators is an appropriate one. By the above lemma, this class is different from the class of the \mathcal{F} -substitutional operators for any choice of the class $\mathcal{F} \subseteq \mathbb{T}$. There are, however, more interesting examples of appropriate classes with this property. Such classes are, for instance, the class of all computable operators, the class of the primitive recursive ones, the class of the elementary ones, etc. Although consisting of continuous operators, each of these classes contains some operators which are not \mathcal{F} -substitutional, whatever be the class \mathcal{F} . This can be seen by inductively proving the stronger continuity property of the \mathcal{F} -substitutional operators formulated in the lemma below.

Lemma 2.15. *Let \mathcal{F} be an arbitrary subclass of \mathbb{T} . Then, whenever F is a k -ary \mathcal{F} -substitutional operator, there exists a natural number v with the following property: for all $f_1, \dots, f_k \in \mathbb{T}_1$ and $n \in \mathbb{N}$, there exists a subset A of \mathbb{N} , such that A has at most v elements, and the equality (2) holds, whenever $g_1, \dots, g_k \in \mathbb{T}_1$ and $g_1(t) = f_1(t), \dots, g_k(t) = f_k(t)$ for all $t \in A$.*

We emphasize that the property from Lemma 2.15 is indeed stronger than ordinary continuity, since the set A may depend on the choice of $f_1, \dots, f_k \in \mathbb{T}_1$ and $n \in \mathbb{N}$, but the cardinality bound v for A may not.

3. Uniform and conditional computability of a real function with respect to a class of operators

As in [7], a triple $(f, g, h) \in \mathbb{T}_1^3$ is called to *name* a real number ξ if

$$\left| \frac{f(t) - g(t)}{h(t) + 1} - \xi \right| < \frac{1}{t + 1}$$

for all $t \in \mathbb{N}$.

Definition 3.1. *Let \mathbf{O} be a class of operators, $N \in \mathbb{N}$ and $\theta : D \rightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. The function θ will be called uniformly computable with respect to \mathbf{O} or uniformly \mathbf{O} -computable, for short, if there exist $3N$ -ary operators F, G, H belonging to \mathbf{O} , such that whenever $(\xi_1, \dots, \xi_N) \in D$ and $(f_1, g_1, h_1), \dots, (f_N, g_N, h_N)$ are triples from \mathbb{T}_1^3 naming ξ_1, \dots, ξ_N , respectively, the triple*

$$(F(f_1, g_1, h_1, \dots, f_N, g_N, h_N), G(f_1, g_1, h_1, \dots, f_N, g_N, h_N), H(f_1, g_1, h_1, \dots, f_N, g_N, h_N))$$

names $\theta(\xi_1, \dots, \xi_N)$.

If $\mathcal{F} \subseteq \mathbb{T}$ and \mathbf{O} is the class of all \mathcal{F} -substitutional operators, then a real function is uniformly \mathbf{O} -computable iff it is uniformly \mathcal{F} -computable in the sense of Definition 7 in [7].

Definition 3.2. *Let \mathbf{O} be a class of operators, $N \in \mathbb{N}$ and $\theta : D \rightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. The function θ will be called conditionally computable with respect to \mathbf{O} or conditionally \mathbf{O} -computable, for short, if there exist a $3N$ -ary operator E and $(3N + 1)$ -ary operators F, G, H , such that $E, F, G, H \in \mathbf{O}$ and, whenever $(\xi_1, \dots, \xi_N) \in D$ and $(f_1, g_1, h_1), \dots, (f_N, g_N, h_N)$ are triples from \mathbb{T}_1^3 naming ξ_1, \dots, ξ_N , respectively, the following holds:*

1. *There exists a natural number s satisfying the equality*

$$E(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(s) = 0. \quad (3)$$

2. *For any natural number s satisfying the equality (3), the triple*

$$(F(f_1, g_1, h_1, \dots, f_N, g_N, h_N, \check{s}), G(f_1, g_1, h_1, \dots, f_N, g_N, h_N, \check{s}), \\ H(f_1, g_1, h_1, \dots, f_N, g_N, h_N, \check{s}))$$

$$\text{names } \theta(\xi_1, \dots, \xi_N).$$

By the case $m = 1$ of Lemma 2.4, if $\mathcal{F} \subseteq \mathbb{T}$, and \mathbf{O} is the class of all \mathcal{F} -substitutional operators, then a real function is conditionally \mathbf{O} -computable iff it is conditionally \mathcal{F} -computable in the sense of Definition 2 in [6].

If \mathbf{O} is an appropriate class of operators then all uniformly \mathbf{O} -computable real functions are conditionally \mathbf{O} -computable. Indeed, let \mathbf{O} be an appropriate class of operators and $F^\circ, G^\circ, H^\circ$ be the operators from Definition 3.1 belonging to \mathbf{O} for the N -argument real function θ . Then we can satisfy the requirements from Definition 3.2 by setting

$$\begin{aligned} E(f_1, g_1, h_1, \dots, f_N, g_N, h_N) &= \text{id}_{\mathbb{N}}, \\ F(f_1, g_1, h_1, \dots, f_N, g_N, h_N, e) &= F^\circ(f_1, g_1, h_1, \dots, f_N, g_N, h_N), \\ G(f_1, g_1, h_1, \dots, f_N, g_N, h_N, e) &= G^\circ(f_1, g_1, h_1, \dots, f_N, g_N, h_N), \\ H(f_1, g_1, h_1, \dots, f_N, g_N, h_N, e) &= H^\circ(f_1, g_1, h_1, \dots, f_N, g_N, h_N). \end{aligned}$$

The operator E belongs to \mathbf{O} by statement 1 of Lemma 2.10. By conditions 1 and 3 from Definition 2.1 and by the fact that $F^\circ, G^\circ, H^\circ$ belong to \mathbf{O} , the operators F, G, H also belong to \mathbf{O} .

If \mathbf{O} is a class of recursive operators then obviously all uniformly \mathbf{O} -computable real functions are TTE computable, and it is easy to show the same for the conditionally \mathbf{O} -computable ones.

4. Substitution in conditionally \mathbf{O} -computable real functions

The next theorem generalizes Theorem 1 of [6].

Theorem 4.1. *Let \mathbf{O} be an appropriate class of operators. Let a two-argument \mathbf{O} -representable function C and \mathbf{O} -representable one-argument functions L and R in \mathbb{N} exist such that*

$$\{(u, v) \in \mathbb{N}^2 \mid C(u, v) = 0\} = \{(0, 0)\}, \quad \{(L(s), R(s)) \mid s \in \mathbb{N}\} = \mathbb{N}^2.$$

Then the substitution operation on real functions preserves conditional \mathbf{O} -computability.

Proof. To avoid writing excessively long expressions, we will restrict ourselves to the case of one-argument functions. Let θ_0 and θ_1 be conditionally \mathbf{O} -computable one-argument real functions. We will show the conditional \mathbf{O} -computability of the function θ , defined by

$$\theta(\xi) = \theta_0(\theta_1(\xi)).$$

For $i = 0, 1$, let E_i, F_i, G_i, H_i be operators from \mathbf{O} , such that $\exists s(E_i(f, g, h)(s) = 0)$ and

$$\forall s(E_i(f, g, h)(s) = 0 \Rightarrow (F_i(f, g, h, \check{s}), G_i(f, g, h, \check{s}), H_i(f, g, h, \check{s})) \text{ names } \theta_i(\xi))$$

for any $\xi \in \text{dom}(\theta_i)$ and any triple (f, g, h) naming ξ . We will show that the requirements of Definition 3.2 for the function θ are satisfied through the operators E, F, G, H defined as follows:³

$$\begin{aligned} E(f, g, h)(s) &= C(E_1(f, g, h)(R(s)), E_0(F_1(f, g, h, \dot{R}(\check{s})), G_1(f, g, h, \dot{R}(\check{s})), H_1(f, g, h, \dot{R}(\check{s}))(L(s))), \\ F(f, g, h, e) &= F_0(F_1(f, g, h, \dot{R}(e)), G_1(f, g, h, \dot{R}(e)), H_1(f, g, h, \dot{R}(e)), \dot{L}(e)), \\ G(f, g, h, e) &= G_0(F_1(f, g, h, \dot{R}(e)), G_1(f, g, h, \dot{R}(e)), H_1(f, g, h, \dot{R}(e)), \dot{L}(e)), \\ H(f, g, h, e) &= H_0(F_1(f, g, h, \dot{R}(e)), G_1(f, g, h, \dot{R}(e)), H_1(f, g, h, \dot{R}(e)), \dot{L}(e)). \end{aligned}$$

Suppose $\xi \in \text{dom}(\theta)$ and (f, g, h) is a triple naming ξ . By the conditional **O**-computability of θ_1 , there exists $s_1 \in \mathbb{N}$ such that

$$E_1(f, g, h)(s_1) = 0, \quad (4)$$

and if we choose such an s_1 , then the number $\theta_1(\xi)$ is named by the triple (f_1, g_1, h_1) , where

$$f_1 = F_1(f, g, h, \check{s}_1), \quad g_1 = G_1(f, g, h, \check{s}_1), \quad h_1 = H_1(f, g, h, \check{s}_1). \quad (5)$$

By the conditional **O**-computability of θ_0 , there exists $s_0 \in \mathbb{N}$, such that

$$E_0(f_1, g_1, h_1)(s_0) = 0. \quad (6)$$

If s is a natural number, such that $L(s) = s_0$, $R(s) = s_1$, then $E(f, g, h)(s) = 0$. Consider now any natural number s , such that $E(f, g, h)(s) = 0$. Let $s_0 = L(s)$ and $s_1 = R(s)$. The equality $E(f, g, h)(s) = 0$ implies the equality (4), as well as the equality (6) for the functions f_1, g_1, h_1 , defined by means of the equalities (5). It follows from the equality (4) that (f_1, g_1, h_1) names $\theta_1(\xi)$, and, together with the equality (6), this fact implies that $\theta(\xi) = \theta_0(\theta_1(\xi))$ is named by the triple

$$(F_0(f_1, g_1, h_1, \dot{L}(\check{s})), G_0(f_1, g_1, h_1, \dot{L}(\check{s})), H_0(f_1, g_1, h_1, \dot{L}(\check{s}))),$$

which coincides with the triple

$$(F(f, g, h, \check{s}), G(f, g, h, \check{s}), H(f, g, h, \check{s})).$$

The operators F, G, H belong to **O** by conditions 1 and 3 of Definition 2.1 and the fact that all the operators $F_0, G_0, H_0, F_1, G_1, H_1, \dot{R}, \dot{L}$ belong to **O**. It remains to show that the operator E also belongs to **O**. Let the operators A and B be defined by

$$\begin{aligned} A(f, g, h)(s) &= E_1(f, g, h)(R(s)), \\ B(f, g, h)(s) &= E_0(F_1(f, g, h, \dot{R}(\check{s})), G_1(f, g, h, \dot{R}(\check{s})), H_1(f, g, h, \dot{R}(\check{s}))(L(s))). \end{aligned}$$

Then we have

$$E(f, g, h)(s) = C(A(f, g, h)(s), B(f, g, h)(s)),$$

hence, by the fact that the function C is **O**-representable and by applying statement 3 from Lemma 2.10, it would be sufficient to show that $A, B \in \mathbf{O}$ in order to conclude that $E \in \mathbf{O}$. Let $U : \mathbb{T}_1^3 \rightarrow \mathbb{T}_1$ be defined by

$$U(f, g, h) = R$$

³The function C will be used to model conjunction. The functions L and R will be used for decoding a natural number s , which codes the values of the parameters s_0 and s_1 , corresponding to the two real functions θ_0 and θ_1 , respectively.

for all $f, g, h \in \mathbb{T}_1$ and $V : \mathbb{T}_1^4 \rightarrow \mathbb{T}_1$ be defined by

$$V(f, g, h, e) = L$$

for all $f, g, h, e \in \mathbb{T}_1$. The operators U and V belong to \mathbf{O} thanks to Corollary 2.11 and the fact that R and L are \mathbf{O} -representable. Then we have

$$A(f, g, h)(s) = E_1(f, g, h)(U(f, g, h)(s)),$$

so $A \in \mathbf{O}$ by conditions 2 and 3 from Definition 2.1 and the fact that $E_1, U \in \mathbf{O}$. To show that B also belongs to \mathbf{O} , we note that, for all $f, g, h \in \mathbb{T}_1$ and $s \in \mathbb{N}$,

$$B(f, g, h)(s) = W(f, g, h, \hat{s})(s),$$

where the operator W is defined by

$$W(f, g, h, e)(s) = E_0(F_1(f, g, h, \hat{R}(e)), G_1(f, g, h, \hat{R}(e)), H_1(f, g, h, \hat{R}(e)))(L(s)).$$

Since $L(s) = V(f, g, h, e)(s)$, the operator W belongs to \mathbf{O} by conditions 1, 2 and 3 from Definition 2.1 and the fact that $E_0, F_1, G_1, H_1, \hat{R}, V$ belong to \mathbf{O} . Therefore, by condition 4 from Definition 2.1, the operator B also belongs to \mathbf{O} . \square

5. Local uniform \mathbf{O} -computability of the conditionally \mathbf{O} -computable real functions

Definition 5.1. Let \mathbf{O} be a class of operators, $N \in \mathbb{N}$ and $\theta : D \rightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. The function θ will be called locally uniformly \mathbf{O} -computable, if any point of D has some neighbourhood U , such that the restriction of θ to $D \cap U$ is uniformly \mathbf{O} -computable.

For any $k, c \in \mathbb{N}$, let the function $\mu_{k,c} : \mathbb{N}^2 \rightarrow \mathbb{N}$ be defined as follows:⁴

$$\mu_{k,c}(x, y) = \begin{cases} c & \text{if } x = k, \\ y & \text{otherwise.} \end{cases}$$

The next theorem generalizes Theorem 2 of [6].

Theorem 5.2. Let \mathbf{O} be an appropriate class of continuous operators, and let the functions \check{c} for all $c \in \mathbb{N}$ and the functions $\mu_{k,c}$ for all $k, c \in \mathbb{N}$ be \mathbf{O} -representable. Then all conditionally \mathbf{O} -computable real functions are locally uniformly \mathbf{O} -computable.

Proof. For an arbitrary function $a \in \mathbb{T}_1$ and any $k \in \mathbb{N}$, let the unary operator $^k a$ be defined as follows:

$$^k a(f)(t) = \begin{cases} a(t) & \text{if } t < k, \\ f(t) & \text{otherwise.} \end{cases}$$

We will show by induction on k that $^k a \in \mathbf{O}$. The operator $^0 a$ belongs to \mathbf{O} by clause 1 of Definition 2.1 since $^0 a(f) = f$ for all $f \in \mathbb{T}_1$. Suppose now, by the inductive hypothesis, that $^k a \in \mathbf{O}$ for a certain $k \in \mathbb{N}$. Then $^{k+1} a$ also belongs to \mathbf{O} by the equality

$$^{k+1} a(f)(t) = \mu_{k, a(k)}(t, ^k a(f)(t)),$$

⁴We will use the functions $\mu_{k,c}$ to model a particular nested if-then-else construction.

statements 1 and 3 of Lemma 2.10, the inductive hypothesis and the \mathbf{O} -representability of $\mu_{k,a(k)}$.

Let now $\theta : D \rightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}$, be a conditionally \mathbf{O} -computable real function, and $\xi_0 \in D$ (for the sake of simplicity, we assume additionally that θ is unary). Let $E, F, G, H \in \mathbf{O}$ be witnesses from Definition 3.2 (with $N = 1$). Let (f_0, g_0, h_0) be a triple naming ξ_0 , and let s_0 be a natural number, satisfying the equality $E(f_0, g_0, h_0)(s_0) = 0$. By the continuity of E , we can choose a natural number u , such that $E(f, g, h)(s_0) = 0$, whenever $f, g, h \in \mathbb{T}_1$ and $f(t) = f_0(t)$, $g(t) = g_0(t)$, $h(t) = h_0(t)$ for all $t \leq u$. Let P, Q, R be the following unary operators:

$$P = {}^{u+1}f_0, \quad Q = {}^{u+1}g_0, \quad R = {}^{u+1}h_0.$$

These operators belong to \mathbf{O} and, for any $f, g, h \in \mathbb{T}_1$, the functions $P(f)$, $Q(g)$, $R(h)$ coincide, respectively, with the functions f_0, g_0, h_0 on $\{t \in \mathbb{N} \mid t \leq u\}$ and with the functions f, g, h on $\{t \in \mathbb{N} \mid t > u\}$. We define U as follows:

$$U = \bigcap_{t=0}^u \left\{ \xi \in \mathbb{R} \mid \left| \frac{f_0(t) - g_0(t)}{h_0(t) + 1} - \xi \right| < \frac{1}{t+1} \right\}.$$

Then U is a neighbourhood of ξ_0 , and whenever a triple (f, g, h) names a real number belonging to U , the triple $(P(f), Q(g), R(h))$ also names this number. Now let us define

$$\begin{aligned} F_0(f, g, h) &= F(P(f), Q(g), R(h), \check{s}_0), \\ G_0(f, g, h) &= G(P(f), Q(g), R(h), \check{s}_0), \\ H_0(f, g, h) &= H(P(f), Q(g), R(h), \check{s}_0). \end{aligned}$$

By conditions 1 and 3 of Definition 2.1, Corollary 2.11 and the fact that \check{s}_0 is \mathbf{O} -representable, F_0, G_0, H_0 all belong to \mathbf{O} . Let $\xi \in D \cap U$ and (f, g, h) name ξ . Then $(P(f), Q(g), R(h))$ also names ξ and moreover,

$$E(P(f), Q(g), R(h))(s_0) = 0.$$

It follows that the triple $(F_0(f, g, h), G_0(f, g, h), H_0(f, g, h))$ names $\theta(\xi)$. By Definition 3.1, we obtain that the restriction of θ to $D \cap U$ is uniformly \mathbf{O} -computable. \square

6. Uniform \mathbf{O} -computability of the locally uniformly \mathbf{O} -computable functions with compact domains

For any $K \in \mathbb{N}$, let δ_K be the function from \mathbb{T}_{2K+1} defined as follows:⁵ for all $x_1, y_1, \dots, x_K, y_K, z$ in \mathbb{N} , if $x_i = 0$ for some $i \in \{1, \dots, K\}$ then

$$\delta_K(x_1, y_1, \dots, x_K, y_K, z) = y_i$$

with the least such i , otherwise

$$\delta_K(x_1, y_1, \dots, x_K, y_K, z) = z.$$

In particular,

$$\delta_1(x, y, z) = \begin{cases} y & \text{if } x = 0, \\ z & \text{otherwise.} \end{cases}$$

⁵We will use the functions δ_K to organize a simple bounded search.

Definition 6.1. A class \mathbf{O} of operators will be called decent if \mathbf{O} is appropriate and the functions $\lambda x.x + 1$, $\lambda xy.x \dot{-} y$ and δ_1 are \mathbf{O} -representable.

Lemma 6.2. Let \mathbf{O} be a decent class of operators. Then all constant functions from \mathbb{T}_1 and the functions δ_K , $K = 0, 1, 2, 3, \dots$, are \mathbf{O} -representable.

Proof. By using the \mathbf{O} -representability of $\lambda x.x + 1$ and $\lambda xy.x \dot{-} y$ together with Lemmas 2.8 and 2.9 we easily see that all constant functions from \mathbb{T}_1 are \mathbf{O} -representable. The \mathbf{O} -representability of the functions δ_K follows from the equalities

$$\begin{aligned}\delta_0(z) &= z, \\ \delta_{K+1}(x_1, y_1, x_2, y_2, \dots, x_{K+1}, y_{K+1}, z) &= \delta_1(x_1, y_1, \delta_K(x_2, y_2, \dots, x_{K+1}, y_{K+1}, z))\end{aligned}$$

by induction on K with application of Lemmas 2.8 and 2.9. \square

Corollary 6.3. If \mathbf{O} is a decent class of continuous operators then all conditionally \mathbf{O} -computable real functions are locally uniformly \mathbf{O} -computable.

Proof. If \mathbf{O} is a decent class of operators then, by the above lemma, the equality

$$\mu_{k,c}(x, y) = \delta_1(x \dot{-} k, \delta_1(k \dot{-} x, c, y), y)$$

and Lemmas 2.8, 2.9, the functions listed in the premise of Theorem 5.2 are \mathbf{O} -representable. \square

The next theorem generalizes and somewhat strengthens Theorem 3 of [6].

Theorem 6.4. Let \mathbf{O} be a decent class of operators. Then all locally uniformly \mathbf{O} -computable real functions with compact domains are uniformly \mathbf{O} -computable.⁶

Proof. Suppose $N \in \mathbb{N}$, $\theta : D \rightarrow \mathbb{R}$, where D is a compact subset of \mathbb{R}^N , and θ is locally uniformly \mathbf{O} -computable. Then there exist $K \in \mathbb{N}$, rational numbers a_{ij} ($i = 1, \dots, K$, $j = 1, \dots, N$) and positive rational numbers d_1, \dots, d_K , such that $D \subseteq U_1 \cup \dots \cup U_K$, where, for $i = 1, \dots, K$,

$$U_i = \{ (\xi_1, \dots, \xi_N) \in \mathbb{R}^N \mid |\xi_1 - a_{i1}| < d_i, \dots, |\xi_N - a_{iN}| < d_i \}$$

and the restriction of θ to $D \cap U_i$ is uniformly \mathbf{O} -computable. We will prove that θ is also uniformly \mathbf{O} -computable. (Of course, the case $K < 2$ is trivial, so we may assume that $K \geq 2$.) In order to prove the uniform \mathbf{O} -computability of θ , we consider the continuous function

$$\rho(\xi_1, \dots, \xi_N) = \max_{i=1, \dots, K} (d_i - \max_{j=1, \dots, N} |\xi_j - a_{ij}|).$$

Since $\rho(\bar{\xi}) > 0$ for all $\bar{\xi} \in D$, there exists a natural number k , such that $\rho(\bar{\xi}) \geq \frac{2}{k+1}$ for any $\bar{\xi} \in D$. For such a k , as it is easy to see, whenever $(\xi_1, \dots, \xi_N) \in D$ and x_1, \dots, x_N are rational numbers satisfying the inequalities $|x_j - \xi_j| < \frac{1}{k+1}$ ($j = 1, \dots, N$), at least one of the numbers

$$r_1 = d_1 - \max_{j=1, \dots, N} |x_j - a_{1j}|, \dots, r_K = d_K - \max_{j=1, \dots, N} |x_j - a_{Kj}|$$

⁶The conclusion of theorem 3 in [6] is equivalent to the particular instance of the present statement for the case when \mathbf{O} is the class of the \mathcal{F} -substitutional operators for a class \mathcal{F} satisfying the conditions of the theorem in question. The class \mathbf{O} in that case will be surely decent by Lemmas 2.3, 2.7 and the equality $\delta_1(x, y, z) = y(1 \dot{-} x) + z(1 \dot{-} (1 \dot{-} x))$.

will be greater than $\frac{1}{k+1}$, and (ξ_1, \dots, ξ_N) will belong to U_i for any $i \in \{1, \dots, K\}$, such that $r_i > \frac{1}{k+1}$. In particular, that will be the case, whenever $(\xi_1, \dots, \xi_N) \in D$, $(f_1, g_1, h_1), \dots, (f_N, g_N, h_N)$ are triples naming ξ_1, \dots, ξ_N , respectively, and

$$x_j = \frac{f_j(k) - g_j(k)}{h_j(k) + 1}, \quad j = 1, \dots, N.$$

For all $i = 1, \dots, K$, let us choose operators $F_i, G_i, H_i \in \mathbf{O}$, according to Definition 3.1, applied for the restriction of θ to $D \cap U_i$ (which is uniformly \mathbf{O} -computable). We define $3N$ -ary operators F, G, H by

$$\begin{aligned} F(f_1, g_1, h_1, \dots, f_N, g_N, h_N) &= F_l(f_1, g_1, h_1, \dots, f_N, g_N, h_N), \\ G(f_1, g_1, h_1, \dots, f_N, g_N, h_N) &= G_l(f_1, g_1, h_1, \dots, f_N, g_N, h_N), \\ H(f_1, g_1, h_1, \dots, f_N, g_N, h_N) &= H_l(f_1, g_1, h_1, \dots, f_N, g_N, h_N), \end{aligned}$$

where l is the least of the numbers $i \in \{1, \dots, K\}$ satisfying the inequality

$$d_i - \max_{j=1, \dots, N} \left| \frac{f_j(k) - g_j(k)}{h_j(k) + 1} - a_{ij} \right| > \frac{1}{k+1}, \quad (7)$$

if there exists such an i , and

$$F(f_1, g_1, h_1, \dots, f_N, g_N, h_N) = G(f_1, g_1, h_1, \dots, f_N, g_N, h_N) = H(f_1, g_1, h_1, \dots, f_N, g_N, h_N) = \check{0},$$

otherwise.

The above reasoning will show that F, G, H are witnesses for the uniform \mathbf{O} -computability of θ , if we succeed to prove that they belong to \mathbf{O} . Of course, the inequality (7) is equivalent to

$$\max_{j=1, \dots, N} \left| \frac{f_j(k) - g_j(k)}{h_j(k) + 1} - a_{ij} \right| < d_i - \frac{1}{k+1}. \quad (8)$$

We will prove the following auxiliary statement.

For any rational numbers a_1, \dots, a_N, q , there exists an \mathbf{O} -representable function $e \in \mathbb{T}_{3N}$ such that, for all $x_1, y_1, z_1, \dots, x_N, y_N, z_N \in \mathbb{N}$, the equivalence

$$e(x_1, y_1, z_1, \dots, x_N, y_N, z_N) = 0 \Leftrightarrow \max_{j=1, \dots, N} \left| \frac{x_j - y_j}{z_j + 1} - a_j \right| < q$$

holds.

Then, for any $i \in \{1, \dots, K\}$, the inequality (8) will be equivalent to some equality of the form

$$e_i(f_1(k), g_1(k), h_1(k), \dots, f_N(k), g_N(k), h_N(k)) = 0$$

with \mathbf{O} -representable e_i . By the definition of the operator F , this will yield the equality

$$\begin{aligned} F(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t) \\ = \delta_K(S_1(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t), F_1(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t), \dots, \\ S_K(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t), F_K(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t), 0), \end{aligned}$$

where

$$S_i(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t) = e_i(f_1(k), g_1(k), h_1(k), \dots, f_N(k), g_N(k), h_N(k)), \quad i = 1, \dots, K.$$

Using Corollary 2.11 and statements 2, 3 in Lemma 2.10, first by the \mathbf{O} -representability of \check{k} and e_1, \dots, e_K we will be able to conclude that $S_1, \dots, S_K \in \mathbf{O}$ and then by the \mathbf{O} -representability of δ_K and $\check{0}$ and the fact that $F_1, \dots, F_K \in \mathbf{O}$, it will be true that $F \in \mathbf{O}$. It could be seen in a similar way that G and H also belong to \mathbf{O} .

To complete the proof, it remains to prove the auxiliary statement. We will firstly prove that, for any rational number a , there exist \mathbf{O} -representable ternary functions lt_a and gt_a such that, for all $x, y, z \in \mathbb{N}$, the equivalences

$$\text{lt}_a(x, y, z) > 0 \Leftrightarrow \frac{x-y}{z+1} < a, \quad \text{gt}_a(x, y, z) > 0 \Leftrightarrow \frac{x-y}{z+1} > a$$

hold. Actually, it is sufficient to show how to construct the function lt_a , since then we may set

$$\text{gt}_a(x, y, z) = \text{lt}_{-a}(y, x, z).$$

If $a = 0$ then the inequality

$$\frac{x-y}{z+1} < a \tag{9}$$

is equivalent to $x < y$ and we may set

$$\text{lt}_a(x, y, z) = y \dot{-} x.$$

To settle the case when $a \neq 0$, we will first construct, for any positive integers b, c , an \mathbf{O} -representable function $\gamma_{b,c} : \mathbb{N}^{b+c} \rightarrow \mathbb{N}$ such that, for all $x_1, \dots, x_b, y_1, \dots, y_c \in \mathbb{N}$,

$$\gamma_{b,c}(x_1, \dots, x_b, y_1, \dots, y_c) > 0 \Leftrightarrow x_1 + \dots + x_b > y_1 + \dots + y_c. \tag{10}$$

The construction is by the following inductive definition, where b and c can be arbitrary positive integers:

$$\begin{aligned} \gamma_{1,1}(x_1, y_1) &= x_1 \dot{-} y_1, \\ \gamma_{1,c+1}(x_1, y_1, \dots, y_c, y_{c+1}) &= \gamma_{1,c}(x_1 \dot{-} y_{c+1}, y_1, \dots, y_c), \\ \gamma_{b+1,c+1}(x_1, \dots, x_b, x_{b+1}, y_1, \dots, y_c, y_{c+1}) &= \delta_1(x_{b+1} \dot{-} y_{c+1}, \\ &\quad \gamma_{b,c+1}(x_1, \dots, x_b, y_1, \dots, y_c, y_{c+1} \dot{-} x_{b+1}), \\ &\quad \gamma_{b+1,c}(x_1, \dots, x_b, x_{b+1} \dot{-} y_{c+1}, y_1, \dots, y_c)). \end{aligned}$$

One proves inductively that all functions $\gamma_{b,c}$ satisfy the equivalence (10) and are \mathbf{O} -representable (of course, the \mathbf{O} -representability of $\lambda xy. x \dot{-} y$ and δ_1 , as well as Lemmas 2.8 and 2.9 are used in the proof of the last statement). If $a > 0$ then $a = b/c$ with some positive integers b, c , and (9) is equivalent to the inequality $c.(x - y) < b.(z + 1)$. The last inequality is easily seen to be equivalent to $c.(x \dot{-} y) < b.(z + 1)$. Thus we may set

$$\text{lt}_a(x, y, z) = \gamma_{b,c}(\underbrace{z+1, \dots, z+1}_{b \text{ times}}, \underbrace{x \dot{-} y, \dots, x \dot{-} y}_{c \text{ times}})$$

in this case. Finally, if $a < 0$ then $a = -c/b$ with some positive integers b, c , and (9) is equivalent to the inequality $b.(y - x) > c.(z + 1)$, which, in turn, is equivalent to $b.(y \dot{-} x) > c.(z + 1)$. Therefore we may set

$$\text{It}_a(x, y, z) = \gamma_{b,c}(\underbrace{y \dot{-} x, \dots, y \dot{-} x}_{b \text{ times}}, \underbrace{z + 1, \dots, z + 1}_{c \text{ times}})$$

now. By using the **O**-representability of the functions $\lambda x.x + 1$, $\lambda xy.x \dot{-} y$ and $\gamma_{b,c}$ ($b, c = 1, 2, 3, \dots$), and making use again of Lemmas 2.8 and 2.9, we see that all the functions It_a defined above are **O**-representable.

Suppose now that rational numbers a_1, \dots, a_N, q are given. The inequality in the right-hand side of the equivalence in the auxiliary statement is equivalent to the conjunction of the inequalities

$$gt_{b_j}(x_j, y_j, z_j) > 0, \quad lt_{c_j}(x_j, y_j, z_j) > 0, \quad j = 1, \dots, N,$$

where $b_j = a_j - q$, $c_j = a_j + q$. Let us set

$$u_j(x, y, z) = \delta_1(gt_{b_j}(x, y, z), lt_{c_j}(x, y, z), lt_{c_j}(x, y, z)), \quad j = 1, \dots, N,$$

and then set

$$e(x_1, y_1, z_1, \dots, x_N, y_N, z_N) = \delta_N(u_1(x_1, y_1, z_1), 1, \dots, u_N(x_N, y_N, z_N), 1, 0).$$

It is easy to check the equivalence from the auxiliary statement, and the **O**-representability of the function e follows from the **O**-representability of the functions $gt_{b_j}, lt_{c_j}, \check{1}, \check{0}, \delta_1, \delta_N$ and Lemmas 2.8, 2.9. \square

Corollary 6.5. *If **O** is a decent class of continuous operators, then all conditionally **O**-computable real functions with compact domains are uniformly **O**-computable.*

Proof. By Corollary 6.3 and the above theorem. \square

7. Appendix: Conditional computability of functions in effective metric spaces

The referee of the paper asked if Definitions 3.1, 3.2 and 5.1 can be extended to functions between metric spaces other than reals, for instance using the representation-theoretic approach of TTE [9, Definition 8.1.2].

To do such an extension, we have to consider effective metric spaces $\mathbf{M} = (M, d, A, \alpha)$ and $\mathbf{M}' = (M', d', A', \alpha')$ in the sense of the above-mentioned definition instead of \mathbb{R}^N and \mathbb{R} , respectively. However, we must assume that the domains of α and α' consist of natural numbers rather than strings over an arbitrary finite alphabet – this is needed, since our operators act on functions in \mathbb{N} . An *ordinary name* of an element ξ of M will be, by definition, a total one-argument function f in \mathbb{N} such that $f(t) \in \text{dom}(\alpha)$ and $d(\alpha(f(t)), \xi) < \frac{1}{t+1}$ for any $t \in \mathbb{N}$ (similarly for ordinary names of the elements of M').⁷ The next definition contains analogs of the above-mentioned definitions for the case of effective metric spaces of the above sort.

⁷We add the adjective “ordinary” in front of “name” in order to distinguish the names used here from the Cauchy names used in [9].

Definition 7.1. Let \mathbf{O} be a class of operators, let $\mathbf{M} = (M, d, A, \alpha)$, $\mathbf{M}' = (M', d', A', \alpha')$ be effective metric spaces with $\text{dom}(\alpha), \text{dom}(\alpha') \subseteq \mathbb{N}$, and let $\theta : D \rightarrow M'$, where $D \subseteq M$. The function θ will be called *uniformly computable with respect to \mathbf{O}* or *uniformly \mathbf{O} -computable*, for short, if there exists a unary operator $T \in \mathbf{O}$ such that, whenever $\xi \in D$ and f is an ordinary name of ξ , the function $T(f)$ is an ordinary name of $\theta(\xi)$. The function θ will be called *conditionally computable with respect to \mathbf{O}* or *conditionally \mathbf{O} -computable*, for short, if there exist a unary operator $E \in \mathbf{O}$ and a binary operator $T \in \mathbf{O}$, such that, whenever $\xi \in D$ and f is an ordinary name of ξ , there exists a natural number s satisfying the equality $E(f)(s) = 0$, and the function $T(f, \check{s})$ is an ordinary name of $\theta(\xi)$ for any such s . The function θ will be called *locally uniformly \mathbf{O} -computable*, if any point of D has some neighbourhood U in \mathbf{M} such that the restriction of θ to $D \cap U$ is uniformly \mathbf{O} -computable.

Here are some analogs of Theorems 4.1, 5.2 and 6.4.

Theorem 7.2 (Analog of Theorem 4.1). Let \mathbf{O} be an appropriate class of operators. Let a two-argument \mathbf{O} -representable function C and \mathbf{O} -representable one-argument functions L and R in \mathbb{N} exist such that

$$\{(u, v) \in \mathbb{N}^2 \mid C(u, v) = 0\} = \{(0, 0)\}, \quad \{(L(s), R(s)) \mid s \in \mathbb{N}\} = \mathbb{N}^2.$$

Let $\mathbf{M}_0 = (M_0, d_0, A_0, \alpha_0)$, $\mathbf{M}_1 = (M_1, d_1, A_1, \alpha_1)$, $\mathbf{M}_2 = (M_2, d_2, A_2, \alpha_2)$ be effective metric spaces with $\text{dom}(\alpha_0), \text{dom}(\alpha_1), \text{dom}(\alpha_2) \subseteq \mathbb{N}$, θ_0 be a conditionally \mathbf{O} -computable partial function from \mathbf{M}_1 to \mathbf{M}_0 , and θ_1 be a conditionally \mathbf{O} -computable partial function from \mathbf{M}_2 to \mathbf{M}_1 . Then the composition of θ_0 and θ_1 is also conditionally \mathbf{O} -computable.

Proof. Let θ be the composition of θ_0 and θ_1 , i.e. θ is the partial function from \mathbf{M}_2 to \mathbf{M}_0 defined by $\theta(\xi) = \theta_0(\theta_1(\xi))$. For $i = 0, 1$, let E_i, T_i be operators from \mathbf{O} , such that

$$\exists s(E_i(f)(s) = 0) \ \& \ \forall s((E_i(f)(s) = 0) \Rightarrow T_i(f, \check{s}) \text{ is an ordinary name of } \theta_i(\xi))$$

for any $\xi \in \text{dom}(\theta_i)$ and any ordinary name f of ξ . Let the operators E and T be defined as follows:

$$\begin{aligned} E(f)(s) &= C(E_1(f)(R(s)), E_0(T_1(f, \check{R}(\check{s}))(L(s))), \\ T(f, e) &= T_0(T_1(f, \check{R}(e)), \check{L}(e)). \end{aligned}$$

The reasoning continues in the same manner, as in the proof of Theorem 4.1. It is seen that the operators E and T are witnesses for the conditional \mathbf{O} -computability of θ . \square

Theorem 7.3 (Analog of Theorem 5.2). Let \mathbf{O} be an appropriate class of continuous operators, and let the functions \check{c} for all $c \in \mathbb{N}$ and the functions

$$\mu_{k,c}(x, y) = \begin{cases} c & \text{if } x = k, \\ y & \text{otherwise} \end{cases}$$

for all $k, c \in \mathbb{N}$ be \mathbf{O} -representable. Let $\mathbf{M} = (M, d, A, \alpha)$, $\mathbf{M}' = (M', d', A', \alpha')$ be effective metric spaces with $\text{dom}(\alpha), \text{dom}(\alpha') \subseteq \mathbb{N}$. Then all conditionally \mathbf{O} -computable partial functions from \mathbf{M} to \mathbf{M}' are locally uniformly \mathbf{O} -computable.

Proof. Let $\theta : D \rightarrow M'$, where $D \subseteq M$, be a conditionally \mathbf{O} -computable partial function, and let $\xi_0 \in D$. Let E and T be operators from \mathbf{O} , such that

$$\exists s(E(f)(s) = 0) \ \& \ \forall s(E(f)(s) = 0) \Rightarrow T(f, \check{s}) \text{ is an ordinary name of } \theta(\xi)$$

for any $\xi \in \text{dom}(\theta)$ and any ordinary name f of ξ . Let f_0 be an ordinary name of ξ_0 , and let s_0 be a natural number, satisfying the equality $E(f_0)(s_0) = 0$. By the continuity of E , we can choose a natural number u , such that $E(f)(s_0) = 0$, whenever $f \in \mathbb{T}_1$ and $f(t) = f_0(t)$ for all $t \leq u$. Let $P = {}^{u+1}f_0$ (in the notations used in the proof of Theorem 5.2), let U be defined by

$$U = \bigcap_{t=0}^u \left\{ \xi \in M \mid d(\alpha(f_0(t)), \xi) < \frac{1}{t+1} \right\}$$

and let us set

$$T_0(f) = T(P(f), \check{s}_0).$$

Then, as in the proof of Theorem 5.2, U is a neighbourhood of ξ_0 , P belongs to \mathbf{O} and the operator T_0 is a witness for the uniform \mathbf{O} -computability of the restriction of θ to $D \cap U$. \square

Theorem 7.4 (Analog of Theorem 6.4). *Let \mathbf{O} be an appropriate class of operators such that the function δ_1 and all constant functions from \mathbb{T}_1 are \mathbf{O} -representable. Let $\mathbf{M} = (M, d, A, \alpha)$, $\mathbf{M}' = (M', d', A', \alpha')$ be effective metric spaces with $\text{dom}(\alpha), \text{dom}(\alpha') \subseteq \mathbb{N}$, and let, for any a in A and any rational number q , there exists an \mathbf{O} -representable function from \mathbb{T}_1 having the value 0 exactly for those $n \in \text{dom}(\alpha)$ which satisfy the inequality $d(\alpha(n), a) < q$. Then all locally uniformly \mathbf{O} -computable partial functions with compact domains from \mathbf{M} to \mathbf{M}' are uniformly \mathbf{O} -computable.*

Proof. Let $\theta : D \rightarrow \mathbb{M}'$ be a locally uniformly \mathbf{O} -computable partial function from \mathbf{M} to \mathbf{M}' , and let its domain D be compact. Then there exist $K \in \mathbb{N}$, elements a_1, \dots, a_K of A and positive rational numbers d_1, \dots, d_K such that $D \subseteq U_1 \cup \dots \cup U_K$, where, for $i = 1, \dots, K$,

$$U_i = \{ \xi \mid d(\xi, a_i) < d_i \}$$

and the restriction of θ to $D \cap U_i$ is uniformly \mathbf{O} -computable. We will prove that θ is also uniformly \mathbf{O} -computable. (Of course, the case $K < 2$ is trivial, so we may assume that $K \geq 2$.) In order to prove the uniform \mathbf{O} -computability of θ , we consider the continuous function

$$\rho(\xi) = \max_{i=1, \dots, K} (d_i - d(\xi, a_i)).$$

Since $\rho(\xi) > 0$ for all $\xi \in D$, there exists a natural number k , such that $\rho(\xi) \geq \frac{2}{k+1}$ for any $\xi \in D$ and let us choose such a k . For any $\xi \in D$ and any ordinary name f of ξ , as in the proof of Theorem 6.4, at least one of the numbers

$$r_1 = d_1 - d(\alpha(f(k)), a_1), \dots, r_K = d_K - d(\alpha(f(k)), a_K)$$

will be greater than $\frac{1}{k+1}$ and ξ will belong to U_i for any $i \in \{1, \dots, K\}$, such that $r_i > \frac{1}{k+1}$. For any $i \in \{1, \dots, K\}$, let us choose an operator $T_i \in \mathbf{O}$, such that, whenever $\xi \in D \cap U_i$ and f is an ordinary name of ξ , the function $T_i(f)$ is an ordinary name of $\theta(\xi)$ (such operators exist due to

the uniform \mathbf{O} -computability of the restriction of θ to any of the sets $D \cap U_1, \dots, D \cap U_K$. For any such i , let e_i be an \mathbf{O} -representable function from \mathbb{T}_1 such that

$$\forall n \in \text{dom}(\alpha) \left(e_i(n) = 0 \Leftrightarrow d(\alpha(n), a_i) < d_i - \frac{1}{k+1} \right).$$

We define an operator T by setting

$$T(f) = T_l(f),$$

where l is the least of the numbers $i \in \{1, \dots, K\}$ satisfying the equality $e_i(f(k)) = 0$ if there exists such an i , and

$$T(f) = \check{0},$$

otherwise. The above reasoning will show that T is a witness for the uniform \mathbf{O} -computability of θ if we succeed to prove that $T \in \mathbf{O}$. To show this, we note that, for any $f \in \mathbb{T}_1$, the equality

$$T(f)(t) = \delta_K(e_1(f(k)), T_1(f)(t), \dots, e_K(f(k)), T_K(f)(t), 0)$$

holds. As in the proof of Lemma 6.2, we can show that all functions $\delta_1, \delta_2, \delta_3, \dots$ are \mathbf{O} -representable. The \mathbf{O} -representability of the functions $\delta_K, e_1, \dots, e_K, \check{k}$, together with Corollary 2.11 and statements 2, 3 in Lemma 2.10, imply that $T \in \mathbf{O}$. \square

Unfortunately, we need an additional assumption about the class \mathbf{O} for being able to consider Theorems 4.1, 5.2 and 6.4 as particular instances of their analogs proved above. The assumption is, roughly speaking, about the existence of an \mathbf{O} -representable mechanism in \mathbb{N} for coding of ordered pairs and the corresponding decoding. Namely, let \mathbf{O} be an appropriate class of operators, and let there exist an \mathbf{O} -representable function $J \in \mathbb{T}_2$ and \mathbf{O} -representable functions $L, R \in \mathbb{T}_1$ such that $L(J(u, v)) = u$ and $R(J(u, v)) = v$ for all $u, v \in \mathbb{N}$. Then we can turn any \mathbb{R}^N into an effective metric spaces \mathbb{M}_N in such a way that the notions introduced Definitions 3.1, 3.2 and 5.1 are equivalent to the corresponding notions introduced here for partial functions from \mathbb{M}_N to \mathbb{M}_1 .

First of all, the above assumptions imply that, for any positive integer K there exist an \mathbf{O} -representable function $J_K \in \mathbb{T}_K$ and \mathbf{O} -representable functions $P_{K,1}, \dots, P_{K,K} \in \mathbb{T}_1$ such that

$$P_{K,i}(J_K(u_1, \dots, u_K)) = u_i, \quad i = 1, \dots, K,$$

for all $u_1, \dots, u_K \in \mathbb{N}$.

For any positive integer N , we consider the effective metric space $\mathbb{M}_N = (\mathbb{R}^N, d_N, \mathbb{Q}^N, \alpha_N)$, where, as usually, \mathbb{Q} is the set of the rational numbers, and d_N, α_N are defined as follows:

$$d_N((\xi_1, \dots, \xi_N), (\eta_1, \dots, \eta_N)) = \max(|\xi_1 - \eta_1|, \dots, |\xi_N - \eta_N|),$$

$$\alpha_N(n) = \left(\frac{P_{3N,1}(n) - P_{3N,2}(n)}{P_{3N,3}(n) + 1}, \dots, \frac{P_{3N,3N-2}(n) - P_{3N,3N-1}(n)}{P_{3N,3N}(n) + 1} \right).$$

It is easy to prove that a partial function from \mathbb{R}^N to \mathbb{R} is uniformly or conditionally \mathbf{O} -computable if and only if it is uniformly or conditionally \mathbf{O} -computable, respectively, as a partial function from \mathbb{M}_N to \mathbb{M}_1 (after the identification of \mathbb{R}^1 and \mathbb{Q}^1 with \mathbb{R} and \mathbb{Q} , respectively). Namely, if $\theta : D \rightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^N$, then:

1. If F, G, H have the properties from Definition 3.1, then the operator T defined by

$$T(f) = \check{J}_3(F(P_{3N,1}^\circ(f), P_{3N,2}^\circ(f), P_{3N,3}^\circ(f), \dots, P_{3N,3N-2}^\circ(f), P_{3N,3N-1}^\circ(f), P_{3N,3N}^\circ(f)),$$

$$G(P_{3N,1}^\circ(f), P_{3N,2}^\circ(f), P_{3N,3}^\circ(f), \dots, P_{3N,3N-2}^\circ(f), P_{3N,3N-1}^\circ(f), P_{3N,3N}^\circ(f)),$$

$$H(P_{3N,1}^\circ(f), P_{3N,2}^\circ(f), P_{3N,3}^\circ(f), \dots, P_{3N,3N-2}^\circ(f), P_{3N,3N-1}^\circ(f), P_{3N,3N}^\circ(f)))$$

is a witness for the \mathbf{O} -computability of θ as a partial function from \mathbb{M}_N to \mathbb{M}_1 .

2. If T is a witness for the \mathbf{O} -computability of θ as a partial function from \mathbb{M}_N to \mathbb{M}_1 , then the operators F, G, H defined by

$$\begin{aligned} F(f_1, g_1, h_1, \dots, f_N, g_N, h_N) &= P_{3,1}^\circ(T(J_{3N}^\circ(f_1, g_1, h_1, \dots, f_N, g_N, h_N))), \\ G(f_1, g_1, h_1, \dots, f_N, g_N, h_N) &= P_{3,2}^\circ(T(J_{3N}^\circ(f_1, g_1, h_1, \dots, f_N, g_N, h_N))), \\ H(f_1, g_1, h_1, \dots, f_N, g_N, h_N) &= P_{3,3}^\circ(T(J_{3N}^\circ(f_1, g_1, h_1, \dots, f_N, g_N, h_N))) \end{aligned}$$

have the properties from Definition 3.1.

3. If the operators E, F, G, H have the properties from Definition 3.2, then the operators E' and T defined by

$$\begin{aligned} E'(f) &= E(P_{3N,1}^\circ(f), P_{3N,2}^\circ(f), P_{3N,3}^\circ(f), \dots, P_{3N,3N-2}^\circ(f), P_{3N,3N-1}^\circ(f), P_{3N,3N}^\circ(f)), \\ T(f, e) &= J_3^\circ(F(P_{3N,1}^\circ(f), P_{3N,2}^\circ(f), P_{3N,3}^\circ(f), \dots, P_{3N,3N-2}^\circ(f), P_{3N,3N-1}^\circ(f), P_{3N,3N}^\circ(f), e), \\ &\quad G(P_{3N,1}^\circ(f), P_{3N,2}^\circ(f), P_{3N,3}^\circ(f), \dots, P_{3N,3N-2}^\circ(f), P_{3N,3N-1}^\circ(f), P_{3N,3N}^\circ(f), e), \\ &\quad H(P_{3N,1}^\circ(f), P_{3N,2}^\circ(f), P_{3N,3}^\circ(f), \dots, P_{3N,3N-2}^\circ(f), P_{3N,3N-1}^\circ(f), P_{3N,3N}^\circ(f), e)) \end{aligned}$$

are witnesses for the conditional \mathbf{O} -computability of θ as a partial function from \mathbb{M}_N to \mathbb{M}_1 .

4. If the operators E' and T are witnesses for the conditional \mathbf{O} -computability of θ as a partial function from \mathbb{M}_N to \mathbb{M}_1 , then the operators E, F, G, H defined by

$$\begin{aligned} E(f_1, g_1, h_1, \dots, f_N, g_N, h_N) &= E'(J_{3N}^\circ(f_1, g_1, h_1, \dots, f_N, g_N, h_N)), \\ F(f_1, g_1, h_1, \dots, f_N, g_N, h_N, e) &= P_{3,1}^\circ(T(J_{3N}^\circ(f_1, g_1, h_1, \dots, f_N, g_N, h_N), e)), \\ G(f_1, g_1, h_1, \dots, f_N, g_N, h_N, e) &= P_{3,2}^\circ(T(J_{3N}^\circ(f_1, g_1, h_1, \dots, f_N, g_N, h_N), e)), \\ H(f_1, g_1, h_1, \dots, f_N, g_N, h_N, e) &= P_{3,3}^\circ(T(J_{3N}^\circ(f_1, g_1, h_1, \dots, f_N, g_N, h_N), e)) \end{aligned}$$

have the properties from Definition 3.2.

For the derivation of Theorem 4.1 from its analog in the case of substitution in a function of more than one argument, the following statement can be additionally proved and used.

Let an \mathbf{O} -representable function C with the property from Theorem 4.1 exist, and let $\theta_1, \dots, \theta_K$ be conditionally \mathbf{O} -computable partial functions from an effective metric space $\mathbf{M} = (M, d, A, \alpha)$ to \mathbb{M}_1 , where $\text{dom}(\alpha) \subseteq \mathbb{N}$. Let the partial function θ from \mathbf{M} to \mathbb{M}_K be defined by

$$\theta(\xi) = (\theta_1(\xi), \dots, \theta_K(\xi)).$$

Then θ is also conditionally \mathbf{O} -computable.

For the derivation of Theorem 6.4 from its analog, the following statement should be additionally proved.

Let the class \mathbf{O} be decent. Then, for any positive integer N , any a in \mathbb{Q}^N and any rational number q , there exists an \mathbf{O} -representable function from \mathbb{T}_1 having the value 0 exactly for those $n \in \mathbb{N}$ which satisfy the inequality $d_N(\alpha_N(n), a) < q$.

This statement easily follows from the auxiliary statement in the proof of Theorem 6.4.

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